Neural fields and visual texture perception

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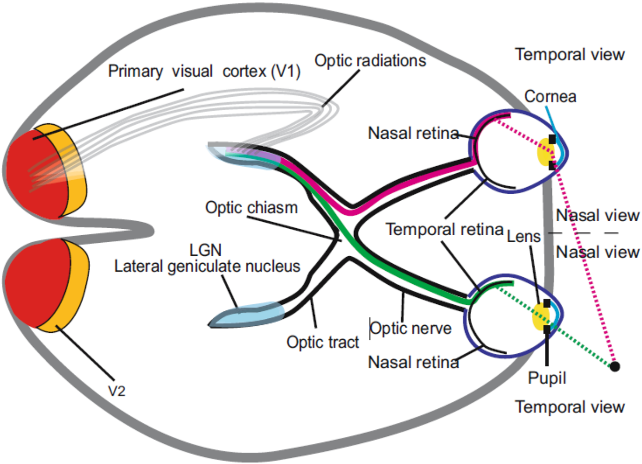
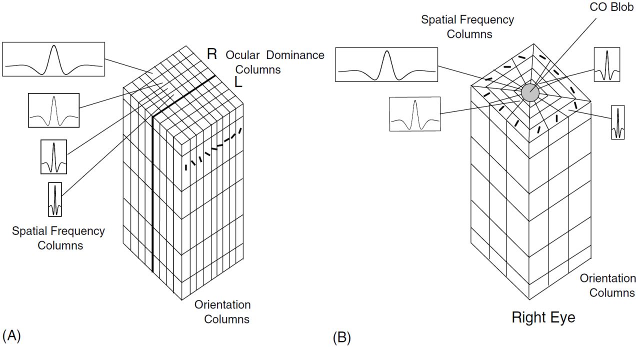
From the retina to V1

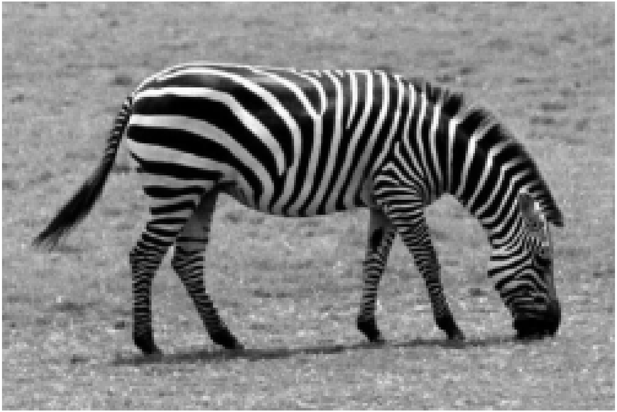


Structure of V1: orientation and scale

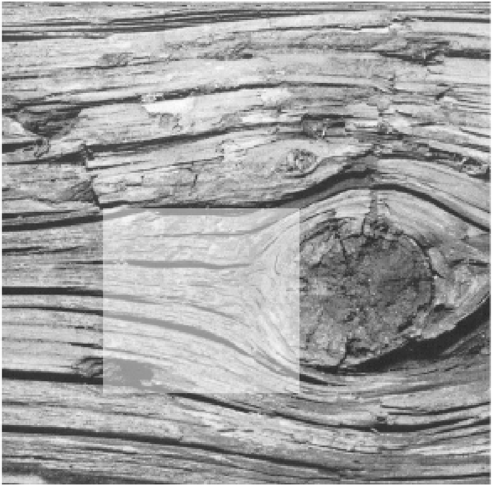
Orientation hypercolumns (Hubel et Wiesel, De Valois):



What is a visual texture?



What is a visual texture?



The structure tensor

- Definition:

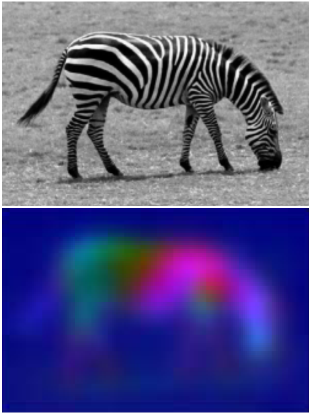
$$J_\sigma = G_\sigma * (\nabla I \nabla I^\top) = \begin{pmatrix} G_\sigma * I_x^2 & G_\sigma * I_x I_y \\ G_\sigma * I_x I_y & G_\sigma * I_y^2 \end{pmatrix}$$

- The variance σ controls the spatial scale.
- This symmetric positive matrix "lives" in a hyperbolic space (Riemann, 1854, Poincaré, 1882).

Image interpretation

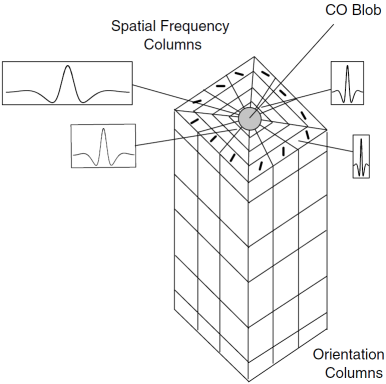
Let $\mathbf{e}_1$ et $\mathbf{e}_2$ be the two orthonormal eigenvectors of $\mathcal{T}$ and $\lambda_1 \geq \lambda_2 \geq 0$ the corresponding eigenvalues

- $\lambda_1 = \lambda_2 = 0$: constant intensity image
- $\lambda_1 >> \lambda_2 \simeq 0$: edge in the direction $\mathbf{e}_2$
- $\lambda_1 \geq \lambda_2 >> 0$: corner
- $\lambda_1 - \lambda_2$ increases with texture anisotropy



Neuronal encoding

- If one "reads" the triplets $(\theta, \theta + \pi/4, \theta + \pi/2)$ from a hypercolumn of orientation, one has access to the three components of the structure tensor in a coordinate system rotated by θ.
- The joint activity of the neurons in the hypercolumn coding for these three orientations is a representation of the structure tensor.
- The set of such triplets is a representation of the structure tensor that is approximately invariant to the orientation of the coordinate system.



Mathematical model

The set $\mathrm{SDP}(2, \mathbb{R})$ of 2×2 symmetric positive definite matrixes with real coefficients is Riemannian space of dimension 3 for the distance

$$d_0(\mathcal{T}_1, \mathcal{T}_2) = \| \log \mathcal{T}_1^{-1} \mathcal{T}_2 \|_F = \left(\sum_{i=1,2} \log^2 \lambda_i \right)^{1/2},$$

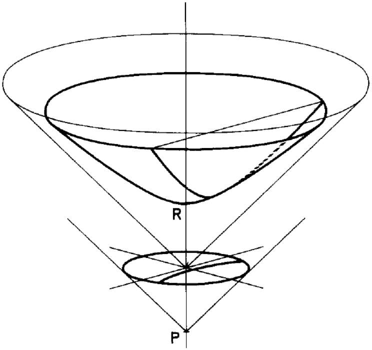
Mathematical model

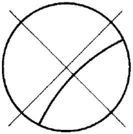
Biological motivation: Unlike the "natural" Euclidean distance, this distance is invariant with respect to changes of coordinate systems defined by $M \in \mathrm{GL}(2, \mathbb{R})$:

$$\mathcal{T} \rightarrow {}^t M \mathcal{T} M$$

Mathematical model

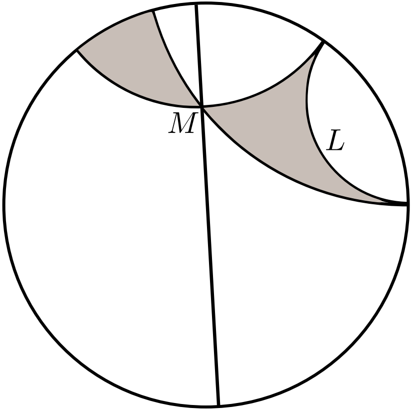
- $\mathrm{SDP}(2,\mathbb{R}) = \mathrm{SSDP}(2,\mathbb{R}) \times \mathbb{R}^+$, where $\mathrm{SSDP}(2,\mathbb{R})$ is the set of symmetric positive definite matrixes with unit determinant.
- $\mathrm{SSDP}(2,\mathbb{R})$ equiped with the Riemannian metric induced by that of $\mathrm{SDP}(2,\mathbb{R})$ has a sectional curvature equal to -1: it is isomorphic to the hyperbolic space of dimension 2, H^2.





Hyperbolic geometry: the Poincaré disk D

The axiom of Euclide: there exists an infinity of lines parallel to L going through the point M



Hyperbolic geometry: the Poincaré disk D

The group of direct isometries

- The group $\mathrm{SU}(1,1)$ of 2×2 Hermitian matrices with unit determinant

$$\gamma = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \text{ such that } |\alpha|^2 - |\beta|^2 = 1,$$

- Its action on D

$$\gamma \cdot z = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}, \quad z \in D$$

- Its action on the structure tensor

$$\tilde{\gamma} \cdot \mathcal{T} = {}^t\tilde{\gamma}\mathcal{T}\tilde{\gamma} \quad \tilde{\gamma} = \begin{bmatrix} \alpha_1 + \beta_1 & \alpha_2 + \beta_2 \\ \beta_2 - \alpha_2 & \alpha_1 - \beta_1 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

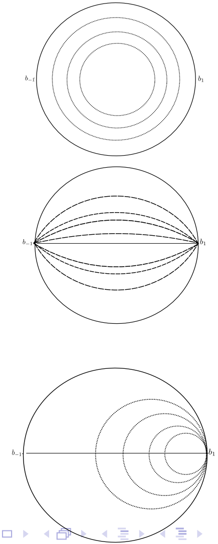
Decomposition of the group of direct isometries

- Three 1-parameter sub-groups of $\mathrm{SU}(1,1)$ and their orbits

$$\begin{cases} K &= \{r_\varphi = \begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix}, \quad \varphi \in S^1\} \\ A &= \{a_t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad t \in \mathbb{R}\} \\ N &= \{n_s = \begin{bmatrix} 1+is & -is \\ is & 1-is \end{bmatrix}, \quad s \in \mathbb{R}\} \end{cases}$$

- Iwazawa decomposition:

$$\mathrm{SU}(1,1) = KAN$$



Decomposition of the group of direct isometries

The same thing for structure tensors

$$\begin{cases} \tilde{r}_\varphi & = \begin{bmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix} \quad \text{Rotation} \\ \tilde{a}_t & = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \quad \text{Scaling} \\ \tilde{n}_s & = \begin{bmatrix} 1 & 0 \\ -2s & 1 \end{bmatrix}, \end{cases}$$

A model of a hypercolumn of structure tensors

- Each structure tensor $\mathcal{T}$ is represented by a population of neurons through its average membrane potential $V(\mathcal{T}, t)$.
- Each population $\mathcal{T}$ excites or inhibits population $\mathcal{T}'$ depending upon whether $\mathcal{T}$ and $\mathcal{T}'$ are close to or far from each other.
- We write a neural mass equation in the Poincaré disk

$$\tau V_t(z,t) = -V(z,t) + \int_D w(z,z') S(\mu V(z',t))\, dz' + I_{\rm ext}(z,t),$$

- The surface element dz' is given by

$$dz' = \frac{dx'\, dy'}{(1-|z'|^2)^2}$$

A model of a hypercolumn of structure tensors

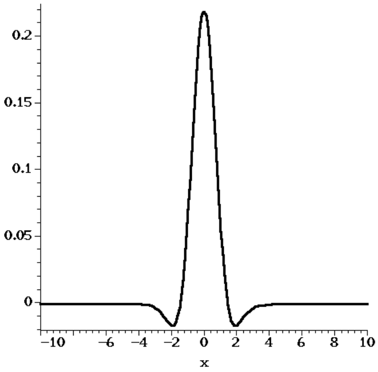
- The connectivity function w is of the form

$$w(z, z') = h(d(z, z')),$$

- where h is a "Mexican hat", the difference of two Gaussians

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x^2}{2\sigma_1^2}} - \theta \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{x^2}{2\sigma_2^2}},$$

$\sigma_1 < \sigma_2$, $\theta \leq 1$



Well-posedness of the problem

- Functional setting $\mathcal{F} = L^{\infty}(\mathbb{D} \times \mathbb{R}_*^+)$
- For some simple hypotheses on w and I there exists a unique solution to the neural mass equation.
- This solution is bounded.

Numerical experiments

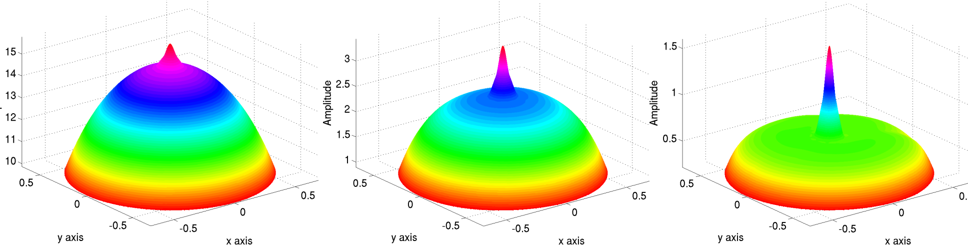
- The neural mass equation is discretised with respect to "space" in a compact domain of $\mathbb{D}$.
- The rectangular rule is used for the integral.
- The numerical scheme is shown to be convergent.

Numerical experiments

Purely excitatory exponential connectivity function $w(x) = e^{-\frac{|x|}{b}}$
$\alpha = 0.1$, $\mu = 10$

Numerical experiments

Constant input $I(z) = \mathcal{I} e^{-\frac{d_2(z,0)^2}{\sigma^2}}$, $\mathcal{I} = 0.1$, $\sigma = 0.05$.

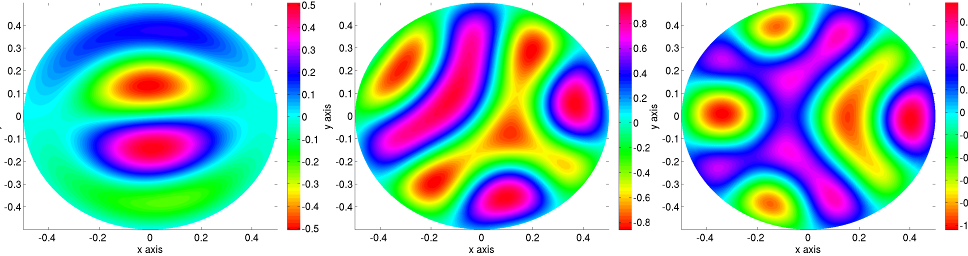


b=1 b=0.2 b=0.1

Numerical experiments

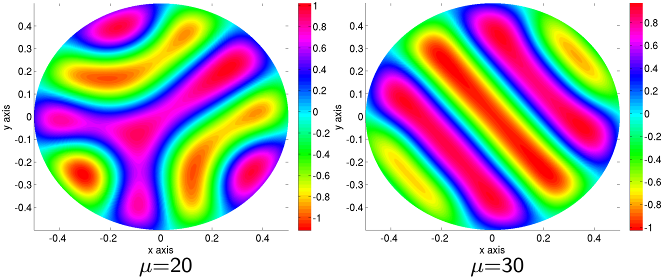
Excitatory and inhibitory connectivity function

$$w(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x^2}{\sigma_1^2}} - \frac{A}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{x^2}{\sigma_2^2}}, \ \sigma_1 = 0.1, \sigma_2 = 0.2 \text{ and } A = 1.$$



μ=3 μ=5 μ=10

Numerical experiments



Periodic pavings of D and discrete subgroups of $\mathrm{SU}(1,1)$

- Discrete subgroups (Fuchsian) Γ such that there exists a closed region F (fundamental domain) of D such that

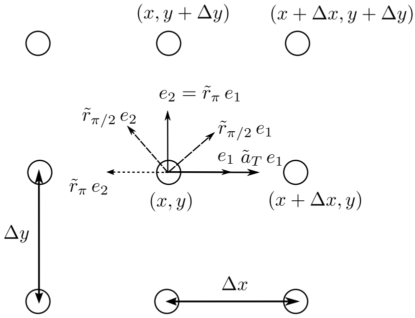
$$\begin{aligned} (i) \quad & \mathring{F} \cap (\gamma \cdot F) = \emptyset \\ & \forall \gamma \in \Gamma,\ \gamma \neq Id \\ (ii) \quad & D = \bigcup_{\gamma \in \Gamma} \gamma \cdot F \end{aligned}$$

- If F is compact, Γ is said to be co-compact.

Work of M. Escher :

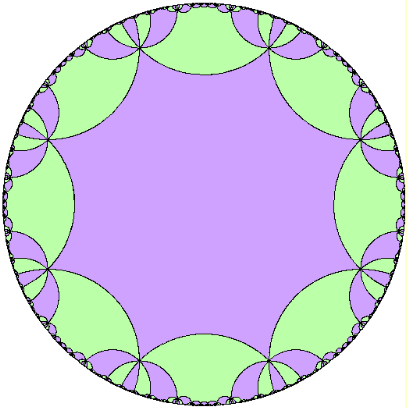
Retinal/Image interpretation

- The neuronal representations of the structure tensor should be invariant with respect to the action of certain discrete subgroups of K (rotations) and A (scalings).
- Fix an integer n (rotation of π/n) and a real T (multiplication of the x coordinate by e^T) and consider the free product $\Gamma_{n,T} = K_n * A_T$.
- It is a "neuronal" Fuchsian group for some values of n and T.



Retinal/Image interpretation

- For $n = 4$ and $\cosh(T) = 1 + \sqrt{2}$, $\Gamma_{n,T}$ is Fuchsian and co-compact.
- Its fundamental domain is included in that of the octogonal Fuchsian group.



The H-planforms

- We study the bifurcations of the solutions when the slope μ of the sigmoid varies.
- In the Euclidean case, one perturbs the solution with planar waves (planforms) of the form $e^{i\mathbf{k}\cdot\mathbf{r}}$, $\mathbf{k} \in \mathbb{R}^2$.
- They are eigenfunctions of the Laplacian operator

$$\Delta e^{i\mathbf{k}\cdot\mathbf{r}} = -\|\mathbf{k}\|^2 e^{i\mathbf{k}\cdot\mathbf{r}}, \ \mathbf{r} \in \mathbb{R}^2.$$

The H-planforms

- It is possible to restrict the problem to a periodic lattice $\mathcal{L}$ generated by two vectors $\mathbf{k}_1$ and $\mathbf{k}_2$.
- The spectrum of Δ is real and discrete on a well-chosen space of periodic functions of $\mathcal{L}$.
- It is the approach of Bressloff et al. to the study of visual hallucinations, see the wonderful book by Jean Petitot.

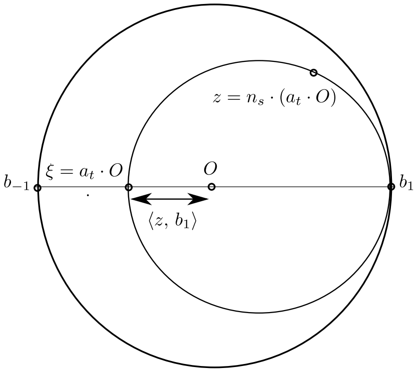
The H-planforms

- Helgason introduced the functions

$$e_{\lambda,b}(z) = e^{(i\lambda+1)\langle z,b\rangle},\ \lambda \in \mathbb{C}$$

- They are eigenfunctions of the Laplace-Beltrami operator in D associated to the eigenvalue $-\lambda^2 - 1$.
- They allow to define a Fourier transform for the functions defined on D.
- An H-planform is a function $e_{\lambda,b}$ for λ real or $\lambda = \alpha + i$, α real.

Horocyclic coordinates:



The H-planforms

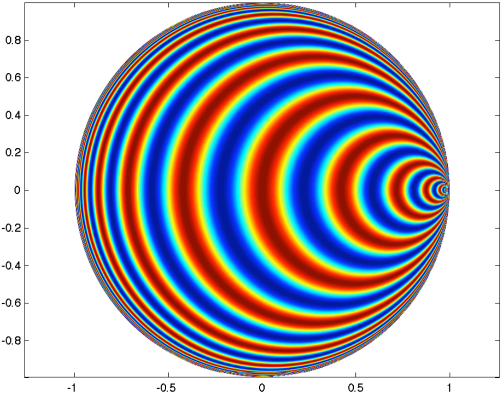
- Invariant with respect to the action of the group N
- Analog to planar waves in the Euclidian plane

In horocyclic coordinates $z = n_s a_t$

$$e_{\lambda, b_1}(z) = e^{(i\lambda+1)t}, \lambda = \alpha + i$$

is periodic with period $2\pi/\alpha$ with respect to t

A periodic H-planform



Bifurcation of the solutions of the structure tensor equation: the search for H-planforms

- The spectrum of the Laplace-Beltrami operator restricted to Γ-invariant functions is discrete and real
- Each square integrable function can be written as a series of eigenfunctions of the operator Δ

$$\Psi_\lambda(z) = \int_{\partial D} e^{(i\lambda+1)\langle z,b\rangle} dT(b),$$

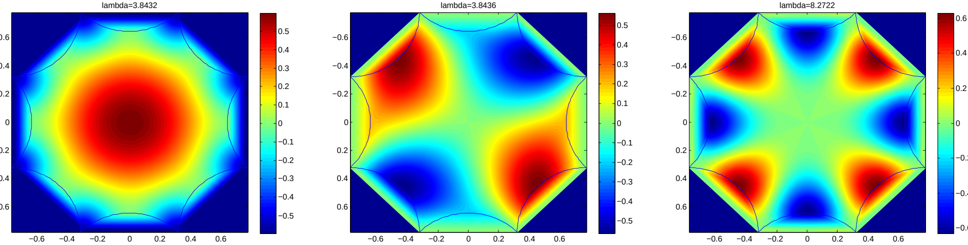
  where T is a distribution on ∂D satisfying some equivariant conditions.
- The values of λ depend upon Γ and there is no explicit method for computing these eigenvalues and the distribution T.
- Some of these eigenfunctions may be observable when the solutions of the structure tensor equation bifurcate.

Toward biological predictions

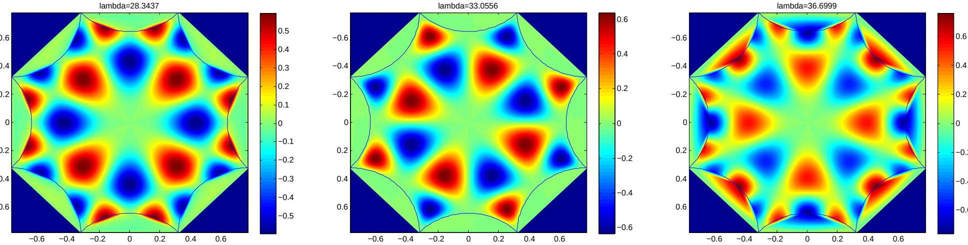
- In the case of the octogonal Fuchsian group some advances have been made (Balazs-Voros, Physics reports, 1986, our current work).
- They lead to the prediction of certain forms of activity.
- They strongly depend upon the type of invariance of the underlying neuronal representations.
- The mathematical theory is a way to test these hypotheses.

Toward biological predictions

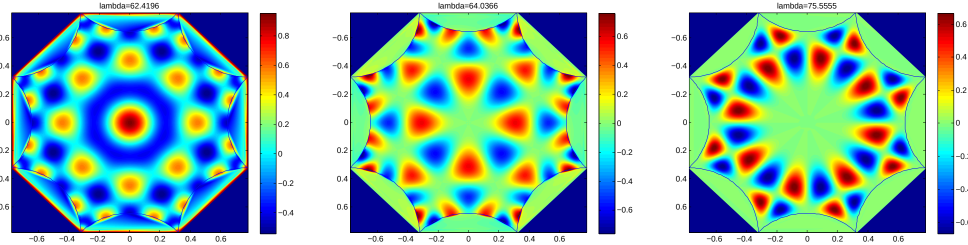
Predicted activity in the case of an invariance with respect to the action of the octogonal group



Toward biological predictions

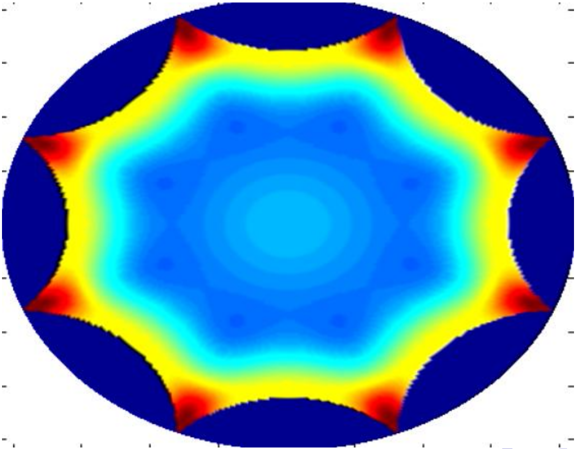


Toward biological predictions



Toward biological predictions

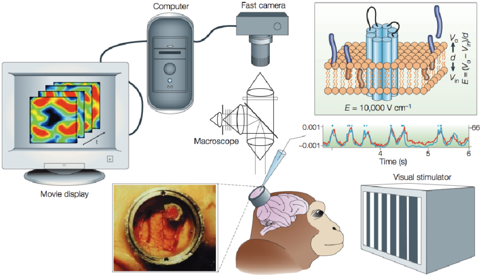
Convergence of the solution of the neural field equation



Biological predictions and mathematical problems

- This example poses difficult mathematical questions, e.g. related to the geometry of "neuronal" Fuchsian groups.
- Mathematical theories lead to precise biological predictions that may be experimentally tested.

Optical imaging principle:



Grinvald-Hildesheim Nature Reviews Neuroscience 04

Generality of the approach

The problem is generic (cortical organisation in columns, excitation/inhibition mechanisms)
Adapted from Chossat and Faugeras, Plos Comp Bio, 2009 plus some recent developments